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COMMENT

Comment on 'Exact evaluation of a path integral relating to an electron gas in a random potential'

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Abstract. We give an alternative derivation of the evaluation by Papadopoulos of the path integral relating to an electron gas. The method employed is a simple application of the powerful technique of global integration on function space. The integration is direct and does not require coupling of the system to auxiliary external forces.

Without loss of generality, we take as our starting point equation (2.2) of Papadopoulos (1974) with $\hbar = m = 1$ and treat the problem in one dimension:

$$G(x,\beta|x_0,0) = \int_{x(0)=x_0}^{x(\beta)=x} \mathscr{D}[x(\tau)] \exp\left(-\int_0^\beta \mathrm{d}\tau(\tfrac{1}{2}\dot{x}^2(\tau)+\tfrac{1}{2}\Omega^2x^2(\tau))\right) \times \exp\left[\frac{\Omega^2}{2\beta}\left(\int_0^\beta x(\tau)\,\mathrm{d}\tau\right)^2\right] \tag{1}$$

$$= G_0(x,\beta|x_0,0)\int_{\Phi_{xx_0}} \exp\left[\frac{\Omega^2}{2\beta}\left(\int_0^\beta x(\tau)\,\mathrm{d}\tau\right)^2\right]\mathrm{d}\omega'(x). \tag{2}$$

Following Maheshwari (1975), we have written the function space integral in (1) as an integral over Φ_{xx_0}, the space of continuous functions $x \in \Phi_{xx_0}$ such that $x(0) = x_0$, $x(\beta) = x$ defined over the interval $T = [0, \beta]$, with a promeasure ω' corresponding to the Uhlenbeck–Ornstein distribution

$$P(x_{t_{k+1}}|x_{t_k}) = \left(\frac{\Omega}{\pi\{1-\exp[-2\Omega(t_{k+1}-t_k)]\}}\right)^{1/2} \times \exp\left(-\Omega\frac{\{x_{t_{k+1}}-x_{t_k}\exp[-\Omega(t_{k+1}-t_k)]\}^2}{1-\exp[-2\Omega(t_{k+1}-t_k)]}\right). \tag{3}$$

$G_0(x,\beta|x_0,0)$ is the propagator of the Bloch equation for the harmonic oscillator. By means of the average trajector $\bar{x}(t)$, Φ_{xx_0} can be mapped onto $\Phi_0(y \in \Phi_0;$ $y(0) = y(\beta) = 0)$ such that

$$x \in \Phi_{xx_0} \to y(t) = x(t) - \bar{x}(t) \in \Phi_0,$$

$$\bar{x}(t) = \int_{\Phi_{xx_0}} x(t)\,\mathrm{d}\omega'(x) = [x\sinh\Omega t + x_0\sinh\Omega(\beta-t)]/\sinh\Omega\beta. \tag{4}$$

The integral in (2) can now be written as an integral on the space Φ_0 with the image ω of promeasure ω':

$$\int_{\Phi_{xx0}} \exp\left[\frac{\Omega^2}{2\beta}\left(\int_0^\beta x(\tau)\,\mathrm{d}\tau\right)^2\right]\mathrm{d}\omega'(x) = \int_{\Phi_0} \exp\left[\frac{\Omega^2}{2\beta}\left(\int_0^\beta (x(\tau)+\bar{x}(\tau))\,\mathrm{d}\tau\right)^2\right]\mathrm{d}\omega(x). \tag{5}$$

The promeasure ω, corresponding to (3), can be defined in terms of its Fourier transform $\mathscr{F}\omega(\mu)$ on the space $\mathscr{M}$ dual of Φ_0, which is the space of bounded measures μ on T, and the covariance $G(r, s)$:

$$\mathscr{F}\omega(\mu) = \exp(-\tfrac{1}{2}W(\mu)),$$

$$W(\mu) = \int_T \mathrm{d}\mu(r)\int_T \mathrm{d}\mu(s)G(r, s),$$

$$G(r, s) = [\theta^-(r-s)\sinh\Omega r\sinh\Omega(\beta-s)+\theta^+(r-s)\sinh\Omega s\sinh\Omega(\beta-r)]/[\Omega\sinh\Omega\beta]. \tag{6}$$

The function space integral on Φ_0 can be readily evaluated by taking a linear map P: $x\in\Phi_0\rightarrow u = \langle\lambda, x\rangle(\equiv\int_0^\beta x(t)\,\mathrm{d}t)\in\mathbb{R}$, where λ is the Lebesgue measure on the interval $T = [0, \beta]$. From Bourbaki (1969) and Maheshwari (1975) one finds that the image of $\mathrm{d}\omega$ under P is

$$\frac{1}{(2\pi W(\lambda))^{1/2}}\exp\left(\frac{-u^2}{2W(\lambda)}\right)\mathrm{d}u$$

on $\mathbb{R}$, where

$$W(\lambda) = \int_T d\lambda(r)\int_T \mathrm{d}\lambda(s)G(r, s) = \frac{1}{\Omega^3}(\beta\Omega-2\tanh\tfrac{1}{2}\Omega\beta). \tag{7}$$

Thus

$$\begin{aligned}\int_{\Phi_0}&\exp\left(\frac{\Omega^2}{2\beta}(\langle\lambda, x\rangle+\langle\lambda, \bar{x}\rangle)^2\right)\mathrm{d}\omega(x)\\ &= \frac{1}{(2\pi W(\lambda))^{1/2}}\int_{-\infty}^{+\infty}\mathrm{d}u\exp\left(\frac{\Omega^2}{2\beta}(u+\langle\lambda, \bar{x}\rangle)^2\right)\exp\left(-\frac{u^2}{2W(\lambda)}\right)\\ &= \left(\frac{\beta\Omega}{2\tanh\frac{1}{2}\Omega\beta}\right)^{1/2}\exp\left(\frac{(x+x_0)^2}{4}\Omega\tanh\frac{\Omega\beta}{2}\right).\end{aligned} \tag{8}$$

Combining equations (2), (5) and (8), we get

$$G(x, \beta|x_0, 0) = \left(\frac{\Omega}{2\pi\sinh\beta\Omega}\right)^{1/2}\left(\frac{\beta\Omega}{2\tanh\frac{1}{2}\beta\Omega}\right)^{1/2}\exp\left(-\frac{\Omega}{4}(x-x_0)^2\coth\frac{\beta\Omega}{2}\right), \tag{9}$$

with the corrected expression of the corresponding result given by Papadopoulos.

References

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